

# A ONE STEP PREDICTION ERROR APPROACH TO THE IDENTIFICATION OF VISCOPLASTIC MATERIAL MODELS

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**Abstract** A new approach to the problem of estimating parameter in material models is presented. The approach is based on a state space representation of the constitutive equations and one step predictions. The differences between one-step predictions and the corresponding measurements are used to design generic one-step prediction error estimators, and in particular, the maximum likelihood method is presented. The one-step predictions are computed through extended Kalman filtering. Consequences of using a time dependent model with least squares regression are analysed. It is shown that if the residuals are a sequence of stochastic variables, correlated with the regressors, the parameter estimates may be biased.

A Monte Carlo study shows that the model parameters of a Norton viscoplastic model are estimated with up to 40% higher precision with the new approach as compared to standard least squares regression. An analysis of the residuals clearly shows that the residuals of the new estimators form an independent sequence of random variables.

**Keywords:** material models, visco-plasticity, identification

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## 1. Introduction

One consequence of the increasing use of complex constitutive equations is the numerous articles which have been published recently on estimation of parameters in material models, see e.g. Courage et al., 1990, Mahnken and Stein, 1996a; Mahnken and Stein, 1996b, Schwertel and Schinke, 1996 and Senseny and Fossum, 1995; Fossum, 1997; Fossum, 1998. Most of these articles are based on statistical methods, primarily least squares regression, or on methods derived from a minimisation of a cost function of some residuals. The two approaches are similar, especially when quadratic optimisation criteria are used. One advantage of the statistical interpretation of the problem is that it provides an estimate of the goodness of fit in terms of the parameter's covariance. The residuals are defined as the differences between some predicted strain values and the corresponding measured strain values, assuming the stresses, or at least the external loads, are known. Naturally, it is also possible to perform the identification with displacement control, predicting the stresses instead of the strains.

The papers of Mahnken and Stein, 1996a, Schwertel and Schinke, 1996 and Fossum, 1997; Fossum, 1998; Senseny and Fossum, 1995 all treated homogeneous uniaxial conditions. Courage et al., 1990 and Mahnken and Stein, 1996b incorporated finite element calculations in the identification process in order to handle multi dimensional problems. The work of Courage et al., 1990 treat a hyper elastic material while the others consider time dependent materials.

The majority of all estimation techniques relies on some assumptions about the distribution of the data or the parameters. Deviation from this assumed distribution may result in biased estimators, i.e. the mathematical expectation of the parameter estimates is not equal to the true parameter values.

In this work, consequences of using a time dependent model with least squares regression are analysed. It will be shown that when the residuals are a sequence of stochastic variables, correlated with the regressors, the parameter estimates may be biased. An in depth analysis of this problem and ways to work around it is found in the book by Gallant, 1987.

A new approach will be presented, based on a state space representation of the constitutive equations and one step predictions. It is inspired by results from time series analysis, see Madsen and Holst, 1999, signal processing, see Jazwinski, 1970, and automatic control and identification, see Ljung, 1998. The basic idea is to compute one step predictions of the strain values. The integration process is extended to perform a state update after each integration step. This process is called state filtering, Jazwinski, 1970, and consist of repeating the following three steps. First, from the current state, the constitutive equations are integrated to a new time instant to give the total strain. The pre-

diction error is obtained by comparing the calculated value with the measured quantity. Second, the statistics of the predicted variable are computed using the dynamics of the constitutive model. Third, update the current state of the constitutive model. The updating is based both the predicted and measured values and their statistics.

In practice, the solution of the state filtering problem requires a few approximations. Here, extended Kalman filtering, as presented by Wall and Holst, 2001, will be used. It is able to produce residuals with approximately white noise properties. From the residuals, generic one-step prediction error estimators are derived. In particular, the maximum likelihood method will be presented.

The new approach will be compared with the least squares regression method by Monte Carlo simulations.

## Notations

A few notations will be introduced first. The measurements of strains are taken at discrete time instances  $t_k$ ,  $k \in \{0, 1, \dots, N\}$ . A subscript will denote the time  $t_k$  of observation,  $\epsilon(t_k) = \epsilon_k$ . A subset of the observations is  $\epsilon^j = \{\epsilon_j, \epsilon_{j-1}, \dots, \epsilon_1, \epsilon_0\}$  and  $\epsilon^N$  denotes the entire measurement set. The expectation of a variable will be indicated by a hat,  $\hat{a} = E[a]$ . Conditioning of  $a$  on  $b$  will be denoted  $(a|b)$ . The conditional expectation of  $\epsilon_k$  on  $\epsilon^{k-1}$  is denoted  $\hat{\epsilon}_{k|k-1}$ .

## 2. Least squares regression

When using time dependent constitutive equations, the mathematical formulation consists of differential equations. After employing a numerical method to integrate these equations from time  $t_{k-1}$  to time  $t_k$ , they can be written as

$$\epsilon_k = F(\sigma_k, \epsilon_{k-1}, \boldsymbol{\theta}) \quad (1)$$

with  $\epsilon_k$  denoting the total strain at time  $t_k$ ,  $\sigma$  is the stress and  $\boldsymbol{\theta}$  is a vector containing the parameters of the material model. As  $\epsilon_k$  is the variable which is calculated from the model, it is also called the dependent variable.

In order to illustrate how disturbances may effect a least squares regression estimator, it is assumed that the model is studied in an interval where a linearisation is appropriate. A suitable representation for regression is then

$$\epsilon_k = [\sigma_k, \epsilon_{k-1}][\theta_1, \theta_2]^T + w_k \quad (2)$$

where the disturbance  $w_k$  has been introduced to represent both model and measurement errors. The strain recordings, the stresses and the disturbances

are collected in matrices

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \sigma_1 & \epsilon_0 \\ \sigma_2 & \epsilon_1 \\ \vdots & \vdots \\ \sigma_N & \epsilon_{N-1} \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}. \quad (3)$$

Here  $\mathbf{R}$  denote the regressors. The least squares estimator is then

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + \left( \frac{1}{N} \mathbf{R}^T \mathbf{R} \right)^{-1} \left( \frac{1}{N} \mathbf{R}^T \mathbf{W} \right) \quad (4)$$

with  $\boldsymbol{\theta}_0$  denoting the true parameter value, i.e. the parameters which are generating the measured strain values. The estimator is unbiased if the expected value

$$E \left[ \left( \frac{1}{N} \mathbf{R}^T \mathbf{R} \right)^{-1} \left( \frac{1}{N} \mathbf{R}^T \mathbf{W} \right) \right] = 0. \quad (5)$$

This holds if either the residuals  $w_k$  are a sequence of independent stochastic variables or if the regressors are uncorrelated with the residuals. However, as the value of strain at time  $t_k$  depends on  $\epsilon_{k-1}$  in time dependent models, the residual at  $t_k$  depends on  $\epsilon_{k-1}$ . The expectation expressed in equation (5) is therefore not equal to zero and the parameter estimate will therefore be biased.

For problems with dependent residuals with stationary stochastic properties, Gallant, 1987 suggest a transformation of the variables to avoid the bias problem.

### 3. One step predictions

In order to account for model uncertainties and measurement noise, the constitutive equations will be written as a system of stochastic differential equation on state space form

$$\begin{aligned} d\boldsymbol{\epsilon}^{ie} &= f(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^{ie}, \boldsymbol{\theta}, t) dt + d\mathbf{w} & \boldsymbol{\epsilon}^{ie}(t_0) &\in \mathcal{N}(\bar{\boldsymbol{\epsilon}}_{t_0}^{ie}, \mathbf{P}_{t_0}), \\ \boldsymbol{\epsilon}_k &= \mathbf{M}\boldsymbol{\epsilon}^{ie}(t_k) + \mathbf{C}\boldsymbol{\sigma}(t_k) + \mathbf{v}_k. \end{aligned} \quad (6)$$

Here  $\boldsymbol{\epsilon}^{ie}$  denotes inelastic strain,  $\boldsymbol{\epsilon}$  the total strain,  $\boldsymbol{\sigma}$  the stress and  $\mathbf{C}$  is the flexibility matrix. The formulation reflects the facts that the evolution of plastic strain is a continuous process and the measurements are taken at discrete time instances  $t_k$ . The state variables are written as vector quantities in order to provide a more general framework for the present approach. The plastic strain,  $\boldsymbol{\epsilon}^{ie}$ , can include several components of plastic strain as well as other hardening variables. This requires multiplication of a gain matrix,  $\mathbf{M}$ , in the calculation

of the total strain. The gain matrix may also be useful when treating multi-dimensional problems. Model uncertainties are accounted for by the process noise increments  $d\mathbf{w}$  with  $\mathbf{w}$  being a Wiener process. Measurement noise is modelled by the Gaussian process  $\mathbf{v}$ . The noise properties are defined by the covariances  $\text{cov}[\mathbf{w}_t, \mathbf{w}_\tau] = \mathbf{Q}(\min(t, \tau)) \min(t, \tau)$  and  $\text{cov}[\mathbf{v}_k, \mathbf{v}_k] = \mathbf{R}_0$ .

One step predictions of the strain,  $E[\boldsymbol{\epsilon}_k | B y_{k-1}] = \hat{\boldsymbol{\epsilon}}_{k|k-1}$  are calculated through extended Kalman filtering. An algorithm has been derived by Wall and Holst, 2001. The Kalman filter provides means for updating the prediction of strain with feedback information from the measurements, i.e.

$$\hat{\boldsymbol{\epsilon}}_{t_{k+1}|t_{k+1}}^{ie} = \hat{\boldsymbol{\epsilon}}_{t_{k+1}|t_k}^{ie} + \mathbf{K}_{k+1} (\boldsymbol{\epsilon}_{k+1} - \hat{\boldsymbol{\epsilon}}_{k+1|k}). \quad (7)$$

For the details of computing  $\mathbf{K}_{k+1}$ , cf. Wall and Holst, 2001.

### 3.1 Estimators

The one step prediction error, corresponding to the one step predictor is

$$\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta}) = \boldsymbol{\epsilon}_k - \hat{\boldsymbol{\epsilon}}_{k|k-1}. \quad (8)$$

A generic estimator is obtained by minimisation of a cost function  $V(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N)$ , defined as

$$V(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N) = \frac{1}{N} \sum_{k=1}^N l(\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta}), \boldsymbol{\theta}), \quad (9)$$

where  $l(\cdot)$  is a scalar valued, positive function. A generic estimator is

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} V(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N). \quad (10)$$

Under a few regularity assumptions, see Wall and Holst, 2001, the estimate will converge to the true parameter value. According to a central limit theorem, the parameter estimate is a random variable with

$\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \rightarrow \mathcal{N}(0, \mathbf{P}_\theta)$ . The covariance is  $\frac{1}{N}\mathbf{P}_\theta = \text{cov}[\hat{\boldsymbol{\theta}}]$ , with  $\mathbf{P}_\theta$  given by

$$\mathbf{P}_\theta = \left( E \left[ \frac{\partial^2 V(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}^2} \right] \right)^{-1} NE \left[ \left( \frac{\partial V(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} \right) \left( \frac{\partial V(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} \right)^T \right] \left( E \left[ \frac{\partial^2 V(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}^2} \right] \right)^{-1}. \quad (11)$$

**3.1.1 The conditional least squares estimator.** Setting the function  $l(\cdot)$  in (9) equal to  $l(\cdot) = |\cdot|^2$  results in the conditional least squares estimator. The parameter covariance can be calculated from equation (11).

**3.1.2 The maximum likelihood estimator.** The maximum likelihood method is based on formal statistical inference and is a conceptually different approach from minimising prediction errors. The likelihood function is the probability of obtaining the measurements  $p(\boldsymbol{\epsilon}^N; \boldsymbol{\theta})$ . The likelihood function is

$$\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N) = p(\boldsymbol{\epsilon}^N; \boldsymbol{\theta}) = \prod_{k=1}^N p(\boldsymbol{\epsilon}_k | \boldsymbol{\epsilon}^{k-1}; \boldsymbol{\theta}), \quad (12)$$

where successive applications of the rule  $p(a, b) = p(a|b)p(b)$  have been used. The conditional densities  $p(\boldsymbol{\epsilon}_k | \boldsymbol{\epsilon}^{k-1}; \boldsymbol{\theta})$  are recognised as the densities of the residuals,

$$p(\boldsymbol{\epsilon}_k | \boldsymbol{\epsilon}^{k-1}; \boldsymbol{\theta}) = p(\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta}) | \boldsymbol{\epsilon}^{k-1}; \boldsymbol{\theta}) = p(\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta}); \boldsymbol{\theta}). \quad (13)$$

The maximum likelihood estimate is obtained by maximising the likelihood function (12), or equivalently, minimising the negative logarithm likelihood,  $\hat{\boldsymbol{\theta}}_{ML} = \arg \max_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N) = \arg \min_{\boldsymbol{\theta}} (-\log \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^N))$ .

When the residuals are assumed to be stationary Gaussian with covariance matrix  $\mathbf{R}$ , the density function of the residuals is

$$p(\tilde{\boldsymbol{\epsilon}}_k, \boldsymbol{\theta}) = \frac{1}{(2\pi)^{m_\epsilon/2} \sqrt{\det \mathbf{R}}} \exp\left(-\frac{1}{2} \tilde{\boldsymbol{\epsilon}}_k^T(\boldsymbol{\theta}) \mathbf{R}^{-1} \tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta})\right), \quad (14)$$

where  $m_\epsilon$  is the dimension of  $\tilde{\boldsymbol{\epsilon}}_k$ . By using the asymptotic relation  $1/N \sum_k \tilde{\boldsymbol{\epsilon}}_k \tilde{\boldsymbol{\epsilon}}_k^T \rightarrow \mathbf{R}$ , the log likelihood function can be rewritten

$$-\log \mathcal{L} = \frac{N}{2} \log \det \left( \frac{1}{N} \sum_{k=1}^N \tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\theta}) \tilde{\boldsymbol{\epsilon}}_k^T(\boldsymbol{\theta}) \right) + \frac{Nm_\epsilon}{2} (1 + \log 2\pi) \quad (15)$$

when  $\mathbf{R}$  is unknown. The parameter's covariance is calculated as  $\text{cov}[\hat{\boldsymbol{\theta}}_{ML}] = \mathcal{M}^{-1}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$  with the Fisher information matrix  $\mathcal{M}$  defined by

$$\begin{aligned} \mathcal{M} &= E \left[ \left( \frac{\partial}{\partial \boldsymbol{\theta}} \log p(\boldsymbol{\epsilon}^N; \boldsymbol{\theta}) \right) \left( \frac{\partial}{\partial \boldsymbol{\theta}} \log p(\boldsymbol{\epsilon}^N; \boldsymbol{\theta}) \right)^T \right] \Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\ &= -E \left[ \frac{\partial^2}{\partial \boldsymbol{\theta}^2} \log p(\boldsymbol{\epsilon}^N; \boldsymbol{\theta}) \right] \Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}. \end{aligned} \quad (16)$$

The Cramer-Rao inequality states that the covariance of a parameter estimator is  $\text{cov}[\hat{\boldsymbol{\theta}}] \geq \mathcal{M}^{-1}$ , Ljung, 1998. If the covariance takes the lower bound of the Cramer-Rao inequality, the estimator is said to be efficient. If there exists an unbiased efficient estimator, it is the maximum likelihood estimator, cf. Wall and Holst, 2001. For the stationary Gaussian case, the Fisher information can be computed through equation (15).

#### 4. Results

The theory presented above has been evaluated with simulations of a generalised Norton model suitable for transient loading problems, see Wall, 1998. The primary goal has been to compare the one-step prediction approach to standard regression. A block diagram of the evaluation setup is shown in figure 1.

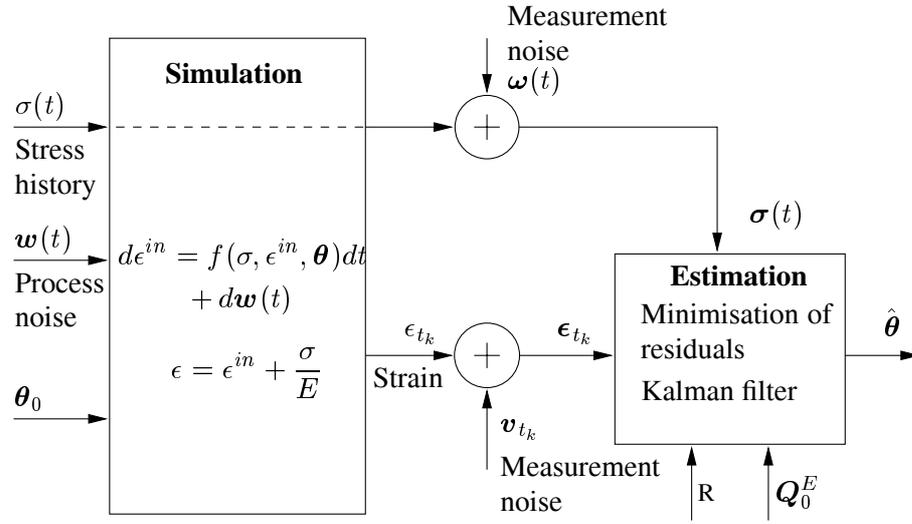


Figure 1. The setup for evaluating the estimator. The estimated parameters  $\hat{\theta}$  should be equal to the true parameters  $\theta_0$  which is used for generation of the data.

The statistics of the added noise processes is given by  $(w_{t_k} - w_{t_{k-1}}) \in \mathcal{N}(0, Q_0)$ ,  $v_k \in \mathcal{N}(0, R_0)$  and  $(\omega_{t_k} - \omega_{t_{k-1}}) \in \mathcal{N}(0, \Omega)$ , as the Wiener processes produces Gaussian increments between two time instances. The outputs from the simulations are then used as inputs to the estimator. The parameter estimator also requires knowledge of the noise properties, which are given by  $Q_0^E$  and  $R_0^E$ . The result is the estimate of the parameters,  $\hat{\theta}$ . The estimator was found to be sensitive to high frequency noise in the stress signal and therefore, it was pre-filtered by a linear phase low pass filter before estimating the parameters.

The solution of the estimation problem is obtained optimisation. The Gauss-Newton method has been used successfully here, cf. Fletcher, 1987 Both the simulation and the estimation require numerical integration. An explicit fifth order Runge-Kutta method was used with a convergence criterion with an absolute tolerance of  $10^{-7}$  for the inelastic strain.

The uniaxial constitutive equation for the Norton model is

$$\dot{\epsilon}^{ie} = f_{\sigma}(\sigma, \epsilon^{ie}, \boldsymbol{\theta}) = \begin{cases} \left( \frac{\sigma - (h(\epsilon^{ie})^n + \sigma_y)}{\eta} \right)^{\gamma} & \text{if } \sigma > (h(\epsilon^{ie})^n + \sigma_y), \\ 0 & \text{if } \sigma \leq (h(\epsilon^{ie})^n + \sigma_y) \end{cases} \quad (17)$$

Isotropic hardening is controlled by the hardening parameters  $h$  and  $n$ . The time dependent, viscous, properties are governed by  $\gamma$  and  $\eta$ . The parameters were chosen to represent a structural steel, with Young's modulus  $E = 206\text{GPa}$ , an initial yield stress  $\sigma_y = 413\text{MPa}$  and the hardening parameters  $n = 0.8$  and  $h = 500\text{MPa}$ . The viscous parameters were  $\gamma_0 = 2$  and  $\eta_0 = 10\text{MPa}$ . Curves of stress-strain without any added noise are shown in figure 2. It can be seen that the model responds nearly rate independent at the lower strain rates. The strain hardening parameters  $h$  and  $n$  can therefore be found by ordinary least squares regression from the low rate response, leaving only the viscous part to be identified by the conditional least squares method. The benefits of splitting the problem into one rate independent, static, part and one viscous part are faster and more robust computations. Hence the unknown parameters are  $\boldsymbol{\theta} = [\gamma \ \eta]$ .

Ljung, 1998 discusses the choice of suitable sampling time, and suggests  $T = t_k - t_{k-1} = -\frac{1}{2} \left( \text{minimum eigenvalue of } \left( \frac{\partial f}{\partial \epsilon^{ie}} \right) \right)^{-1}$ . At the maximum strain rate, which is expected to be at most  $10^3 \text{ s}^{-1}$  and the approximate inelastic strain is 0.1, the sampling time should be less than  $T = (8000)^{-1} \text{ s}$ . The model was simulated at three different strain rates,  $\dot{\epsilon} = [10^1, 10^2, 10^3] \text{ s}^{-1}$ .

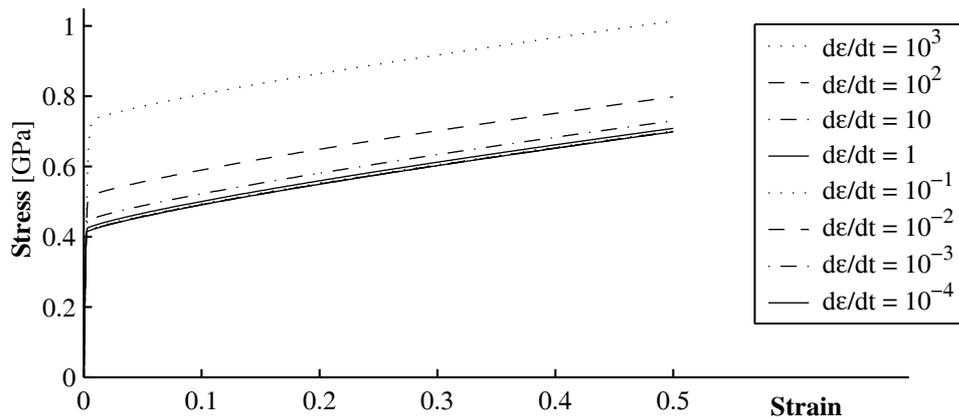


Figure 2. Simulated stress-strain for the generalised Norton model for transient loading.

At the different strain rates, the sampling interval was chosen as  $T = [10^{-4}, 10^{-5}, 10^{-6}]$ s.

A total of 1500 points which were used for each parameter estimate. 200 different parameter estimates were calculated. The mean values and their statistics are shown in table 1. It can be concluded that when fairly correct noise statistics are supplied to the estimator, the conditional least squares estimator has a higher precision than the least squares regression estimator. For the latter, the precision is between 40% to 60% worse. Using the wrong noise statistics in the conditional least squares estimator, as in cases 1 and 2, results in decreasing precision. When noise is added to the input signal, cases 9 and 10, these differences become less pronounced. In the cases of high process noise, cases 7, 8, 15 and 16, the model represents an inaccurate description of the material to be identified. In these cases, the conditional least squares estimator sometimes, about 10% of the investigated samples, failed to find any parameter estimates whereas the regression estimator always succeeded.

A sample of the residuals from one estimation is shown in figure 4 with noise statistics corresponding to case 5. The conditional least squares estimator produces uncorrelated Gaussian random residuals. The residuals from the least squares regression estimator, on the other hand, illustrate that the regression estimator cannot deal with measurements which drift away from the model outputs. Further, the residuals deviates slightly from the Gaussian distribution but the distribution is, however, symmetric. The parameter estimates is therefore probably rather reliable.

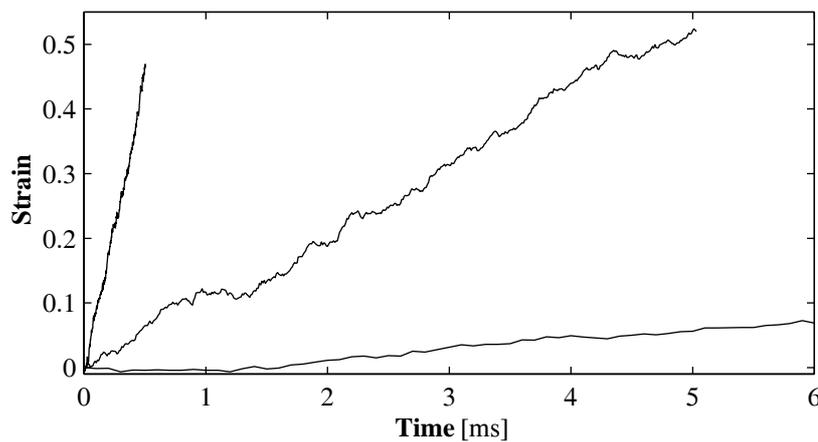


Figure 3. A typical strain output from the simulation of the generalised Norton model for transient loading. The curves represent loading at three different speeds, and only one tenth of the slowest curve is shown. The noise statistics corresponds to case 5 in table 1.

No	Noise characteristics			$\hat{\gamma}_{CLS}$		$\hat{\gamma}_{LSQ}$		$\hat{\eta}_{CLS}$		$\hat{\eta}_{LSQ}$	
	$\sqrt{Q_0^E}$	$\sqrt{Q_0}$	$\sqrt{R_0}$	mean	std	mean	std	mean	std	mean	std
1	0.001	0	0.0025	2.00	0.002	2.00	0.001	9.96	0.03	10.0	0.02
2			0.005	2.00	0.004	2.00	0.002	9.96	0.06	10.0	0.04
3		0.001	0.0025	2.00	0.029	2.00	0.043	10.0	0.37	10.0	0.60
4			0.005	2.00	0.025	2.00	0.040	10.0	0.33	10.0	0.57
5	0.0025	0.0025	0.0025	1.99	0.069	1.99	0.11	9.97	0.86	9.97	1.45
6			0.005	1.99	0.073	1.98	0.11	9.91	0.87	9.80	1.45
7	0.005	0.005	0.0025	1.92	0.27	1.98	0.25	9.89	2.81	9.75	3.34
8			0.005	1.94	0.22	1.94	0.23	9.74	2.38	9.30	2.93
9	0.001	0	0.0025	2.01	0.012	2.00	0.010	10.1	0.19	10.0	0.17
10			0.005	2.00	0.013	2.00	0.010	10.1	0.20	10.0	0.16
11		0.001	0.0025	2.01	0.033	2.00	0.044	10.2	0.44	10.0	0.63
12			0.005	2.01	0.029	2.01	0.041	10.1	0.39	10.1	0.58
13	0.0025	0.0025	0.0025	2.00	0.070	1.98	0.11	10.1	0.89	10.0	1.74
14			0.005	2.00	0.074	1.99	0.11	10.0	0.91	9.83	1.46
15	0.005	0.005	0.0025	1.93	0.26	1.98	0.26	9.89	2.75	9.78	3.35
16			0.005	1.95	0.21	1.95	0.23	9.86	2.30	9.34	2.96

Table 1. The estimates of the parameters for a generalised Norton model for transient loading. The first eight cases were performed with  $\Omega = 0$  and cases 9 to 16 with  $\Omega = (0.02\sigma)^2$ . Each estimation was performed with one time series each at the strain rates  $\dot{\epsilon} = [10, 10^2, 10^3] s^{-1}$ .

## 5. Summary and conclusions

It has been shown how the use of ordinary regression for parameter estimation in dynamic models may result in biased estimates. A one step prediction error approach for the problem of identifying material models has been presented, from which a family of estimators can be derived. The presentation has been accompanied with expressions for the variance-covariance matrices of the estimators. The maximum likelihood estimator is statistically efficient and it offers a flexible framework to handle residuals of any distribution. For Gaussian distributed residuals, the maximum likelihood parameter estimator is similar to the conditional least squares estimates.

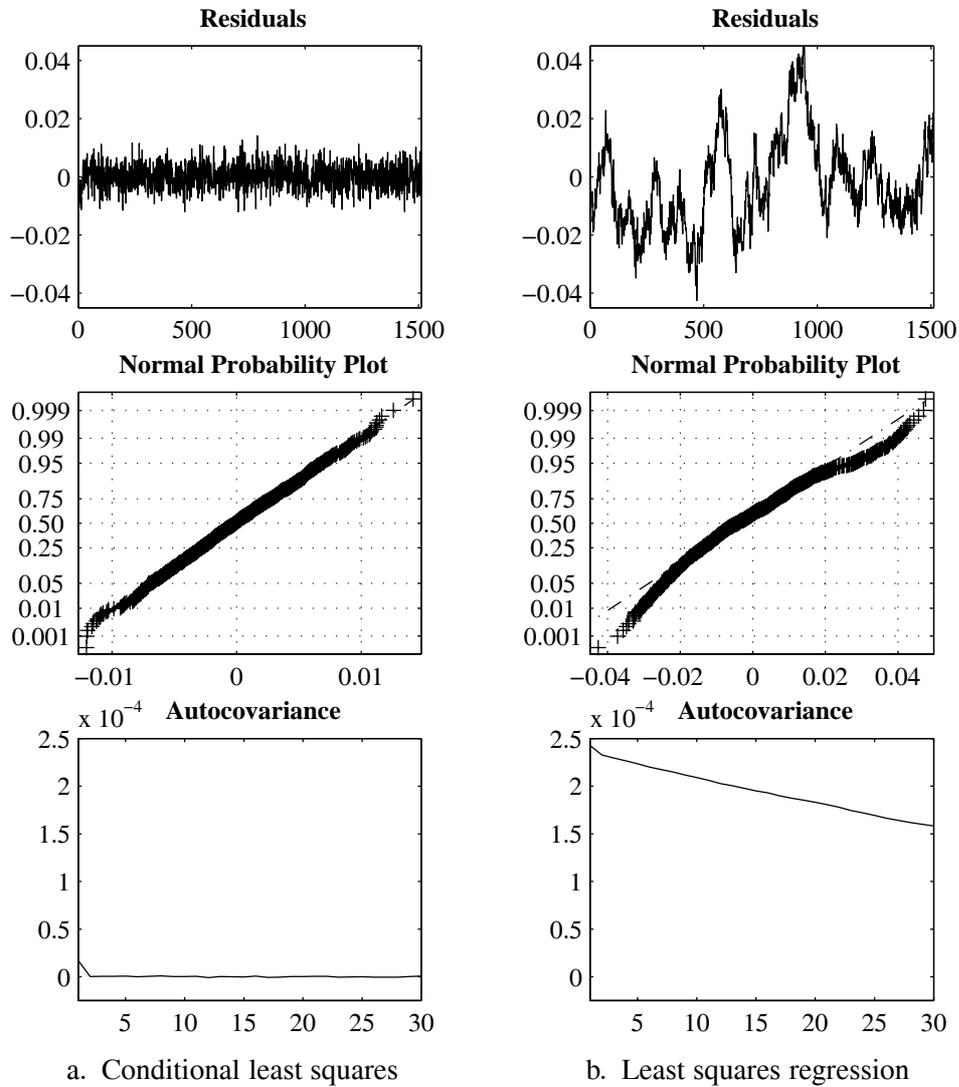


Figure 4. A comparison of the residuals. They were generated with noise statistics corresponding to case 5 in table 1.

A comparison between a traditional least squares regression estimator and the one-step prediction approach showed that the new method produces more accurate estimates and uncorrelated residuals. The precision increased up to 40% when used with a generalised Norton model in transient loading.

The one-step prediction estimators require that the noise statistics is known. The improved precision may be wasted if the wrong noise statistics are sup-

plied to the estimators. In a more advanced implementation, it is possible to estimate the noise statistics simultaneously with the one step predictions, see Madsen and Holst, 1999.

When stress-strain data from a real structural steel were analysed, it turned out that the least squares regression provided more consistent results than the conditional least squares estimator. This was probably caused by inaccurate constitutive equations. The one step prediction approach seems to be sensitive to inaccurate constitutive material models.

Further evaluation of one step prediction error estimators are found in Wall and Holst, 2001, which includes both various levels of excitation as well as models for high temperature creep.

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