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Characterization of strain-induced damage in composites based on the dissipated energy density

Part III. General material constitutive relation

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Abstract

While the energy dissipation function can be approximately determined without explicit knowledge of the damage-induced constitutive nonlinearities as discussed in Parts I and II of this work, a more refined approximation requires a representation of the full-scale nonlinear behavior. This information is also of much interest in its own right since it is needed for stress analysis when analyzing the behavior of structural components loaded well into the nonlinear response range where substantial load redistribution is expected. Part III addresses a representation of the constitutive behavior in terms of the energy dissipation function that is developed with a more refined scheme for identifying the dissipation function itself.

1. Introduction

The basic obstacle to a comprehensive understanding of failure behavior in composites is the complexity of their observed mechanical behavior [1–3]. Composites are generally anisotropic, markedly nonlinear, and, unlike metals, usually fail in an extremely complicated, spatially diffuse, noncatastrophic manner. The nonlinear mechanical behavior of composites is known to be associated with damage accumulation that causes a

local decrease in stiffness of the material in areas where damage is pronounced. Moreover, the complicated manufacturing process results in many imperfections that take on a variety of forms including fiber imperfections, fiber misalignment, and geometrical irregularities in the distribution of fibers, voids or microcracks in the matrix material, and the presence of debonded areas. These defects, combined with high-stress fields near material or geometric discontinuities, induce matrix cracking, fiber breakage, fiber-matrix debonding, and delamination, all of which influence the overall mechanical properties of these materials.

The approximate procedure used in Parts I [4] and II [5] for determining the dissipated energy density function has the following shortcomings:

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first, the required strain fields are computed by assuming linear elastic behavior and by using the elastic constants corresponding to the “virgin” material, rather than by using the proper nonlinear constitutive behavior, and second, the procedure is formulated only in terms of a particular piecewise linear collocation set of basis functions for representing the dissipated energy density function, rather than in terms of arbitrary basis functions. Both of these issues are addressed in Part III such that the underlying assumptions involved in the material model are carefully examined.

2. Basic consideration

To determine the dissipated energy density function, we regard the composite as being composed of either a single mechanically equivalent homogeneous anisotropic material, or a collection of layers of varying orientations of such materials. Provided that the applied loads are either quasi-static or dominated by low temporal frequencies, these homogenization procedures should provide acceptable models since it is expected that the wavelengths corresponding to the spatial variation of the stresses, strains, etc. should be large compared to the microstructural characteristic lengths. If this is not the case, events on the microscale can take on a predominant role and cause significant inaccuracies.

It is postulated that the material constitutive behavior can be completely described by specifying stress as a function of only strain and a set of internal state variables, i.e.,

$$\sigma = C(\xi, \epsilon), \quad (1)$$

where $C(\xi, \epsilon)$ is a nonlinear fourth order tensor function and ξ is the state variable vector. The form of Eq. (1) thus obviously precludes dealing with materials that exhibit marked strain-rate-dependent behavior. Finally, neglected are residual stresses or strains which may exist when the loads are removed such as those induced by the curing process.

The form of the above constitutive relation implies a restricted implicit type of path depen-

dence in the sense that any path dependent behavior results only from internal state transitions that are reflected by changes in the state variables ξ , and not from an explicitly stated load history dependence. This section considers this issue in more detail. It suffices to state here that for the most part the material behaves in a manner similar to that of a hyperelastic material (a nonlinear elastic material with a strain energy potential) as long as the state variables ξ remain unchanged. Thus, for example, when a material point is loaded up to a specified strain, the subsequent structural response is dependent solely on the current strains and is independent of deformation history provided no unloading takes place. Upon unloading, the material behaves elastically in that no further internal material damage takes place; however, a state transition occurs when the unloading commences, and it is this state transition that is responsible for any memory the material might have of its previous loading history. As mentioned in the introduction, this material model and its particular type of path independence is similar to the model employed in [6–9]. Moreover, it is consistent with the observations made in [10] for a type of path independence over a range of tensile and torsional loads in several graphite–epoxy systems.

A general procedure will be developed for determining the dissipation density function using boundary force and displacement data obtained from IPL tests. The procedure is a deconvolution process in the sense that the observations reflect both the effects of material behavior and specimen geometry, and that the geometric effects must be factored out to obtain information relating to the material alone. To be presented is a derivation of the precise analytical form of the constitutive equations.

2.1. Assumptions

A number of assumptions and simplifications have been invoked. While some of which have already been emphasized, a complete account is made as follows:

- The material can be regarded as a mechanically equivalent homogeneous anisotropic mate-

rial as discussed above. In an angle-ply composite, this assumption can be applied either on a ply-by-ply basis or to the collection, as appropriate.

- Loading is either static or slowly varying in accordance with the considerations already discussed.

- The material behavior can be represented as $\sigma = C(\xi, \epsilon)$ discussed previously. This assumption by definition precludes consideration of materials that exhibit marked viscous, rate, and explicit load history dependences. Moreover, given this assumption, a so-called work potential (energy per unit volume), $\psi(\epsilon)$, can be defined such that $\sigma = \text{grad}_\epsilon \psi(\xi, \epsilon)$ [7].

- The total energy absorbed by the material during loading can be regarded as being composed of the sum of a reversible (recoverable) and an irreversible (dissipative) part. The reversible component is the energy that would be recovered if the material were to unload, whereas the irreversible part represents the energy which is dissipated by the internal damage mechanisms. The latter can be described by a dissipation density function $\phi(\epsilon)$ (dissipation energy per unit volume).

- The constitutive relation is continuous in both stress and strain.

- The deformations are sufficiently small so that the infinitesimal stress and strain tensors may be employed.

- Only shell-like structures are considered so that the stress and strain fields in either the entire material or in each ply are two-dimensional with no transverse components, i.e., the stresses and strains can be regarded as vectors having the form $\sigma = (\sigma_{\eta\eta}, \sigma_{\zeta\zeta}, \sigma_{\eta\zeta})^T$, and $\epsilon = (\epsilon_{\eta\eta}, \epsilon_{\zeta\zeta}, \epsilon_{\eta\zeta})^T$, where (η, ζ, ξ) is a coordinate system embedded in the material or layer with one axis along the fiber direction as shown in Fig. 1.

- Displacement continuity is maintained between layers.

It should be stressed here that no explicit assumptions are made concerning the detailed nature of the various failure processes. Rather, the approach is to introduce a minimal number of hypotheses that we feel are in accordance with

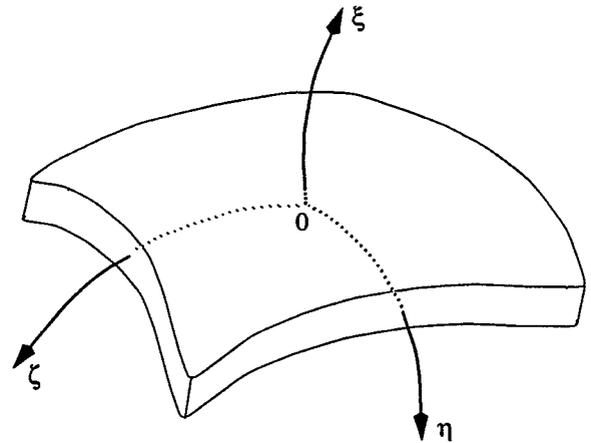


Fig. 1. Coordinate system embedded in the shell structure.

physical fact and readily defensible. Moreover, many of the above restrictions can be relaxed when the situation warrants.

2.2. Analysis overview

The primary issue here is the estimation of the dissipated energy density function $\phi(\epsilon)$ and the subsequent computation of constitutive behavior. We summarize the steps below and provide more detailed explanations in the remaining subsections.

- A representation is chosen for the dissipation function in terms of a set of m basis functions $\chi = (\chi_1, \chi_2, \dots, \chi_m)^T$ and an initially unknown parameter vector $c = (c_1, c_2, \dots, c_m)^T$.

- A uniform set of loading paths in displacement space is selected (15 for each material or 15 for each of a set of layup configurations for angle-ply composites), boundary forces and displacements (f, u) are measured at 50 equally spaced points on each loading path as explained in Part I [4], and the energy imparted to the specimen, W^p , is computed for each observation point p . If the composite is regarded as a single equivalent homogeneous material (in the case of an angle-ply composite, this implies that each layup configuration is to be counted as a separate material), then this procedure yields $n = 750$ values of W^p per material. If the material is an angle-ply composite and we wish to analyze the

constitutive behavior on a ply-by-ply basis, then the procedure yields $n = 750 \times np$ values of W^p for each material, where np denotes the number of layup configurations.

– The irreversible portion of the imparted energy, D^p , is computed for each observation point p using the relationship

$$D^p = W^p - \frac{1}{2}(f^p \cdot u^p), \quad (2)$$

where u^p are the boundary displacements and f^p are the reaction forces.

– The assertion that D^p equals the integral of $\phi(\varepsilon(x))$ over the volume of the specimen, where $x = (\eta, \zeta, \xi)$ is applied at each of the n observation points p and, in conjunction with a representation of the constitutive relation in terms of $\phi(\varepsilon)$ to be described later, the strain-displacement and equilibrium relations, and any appropriate additional constraints, results in a highly overdetermined set of nonlinear equations for the m components of the parameter vector c (a system of $n \times m$ equations with $n \gg m$).

– These equations are solved numerically for c and the dissipation density function $\phi(\varepsilon)$ computed.

– The aforementioned representation of the constitutive relation is used in conjunction with the computed values of c to determine the constitutive behavior $\sigma = C(\xi, \varepsilon)$.

2.3. Dissipation density function

The dissipation density function $\phi(\varepsilon)$ is represented by the linear combination

$$\phi(\varepsilon) = c \cdot \chi(\varepsilon), \quad (3)$$

where $\chi(\varepsilon)$ is a vector of suitable C^0 (or smoother basis) functions over the 3-space $(\varepsilon_{\eta\eta}, \varepsilon_{\zeta\zeta}, \varepsilon_{\eta\zeta})$, and c the parameter vector to be determined. This representation is quite general in the sense that χ_i may be chosen more or less arbitrarily, for instance, as interpolation basis functions over a suitable mesh defined on $(\varepsilon_{\eta\eta}, \varepsilon_{\zeta\zeta}, \varepsilon_{\eta\zeta})$ in which case any of the usual three-dimensional finite element shape functions may be used; as B -splines or cardinal splines; or as locally defined or global orthogonal polynomials. The only significant restriction on the form of the basis functions

is that positivity of the quantity $\phi(\varepsilon)$ must be assured to maintain agreement with physical observation. When χ is an interpolation basis, the components of the parameter vector c are simply the values of ϕ at the node points; however, this is not true when the later two representations for χ are employed. In practice, it is customary to use a locally defined linear interpolation basis which is case in Part I [4].

3. Constitutive relations

An overview of the material model will be given in addition to develop a procedure for computing the constitutive behavior in terms of the dissipated energy density function $\phi(\varepsilon)$. The basic premise is that the material can be regarded as always being in one of two distinct domains: namely, either purely elastic or inelastic. When the material is in an elastic domain, no internal damage is presumed to occur; this results in behavior that is load history independent and reversible in the sense that no energy dissipation takes place. The material enters an inelastic domain during loading after a certain strain-dependent threshold is crossed. Here progressive internal damage leads to behavior that is irreversible in nature with energy being dissipated by the various internal damage mechanisms. A restricted type of load path independence, whose precise nature will be discussed shortly, is applicable in this domain.

One way to model this type of behavior is to postulate the existence of a surface $\Gamma(\xi)$ such as $F(\varepsilon, \xi) = 0$ in strain space with the following property: namely, that whenever the strain ε associated with some material point P is either inside the surface $\Gamma(\xi)$, or outside $\Gamma(\xi)$ with decreasing dissipated energy ($d\phi(\varepsilon)/d|\varepsilon| < 0$), then P is in an elastic domain. When ε is outside of $\Gamma(\xi)$ with increasing dissipated energy ($d\phi(\varepsilon)/d|\varepsilon| > 0$), P is in an inelastic domain. Here the vector ξ is a vector of internal state variables that can be thought of as damage parameters and the surface $\Gamma(\xi)$ acts as a threshold that defines the elastic/inelastic transition. The surface $\Gamma(\xi)$ thus plays a role that is loosely

analogous to that of the yield surface (in stress space) in classical plasticity theory with the state variables ξ in the role of strain hardening parameters.

The model we used is based on the above considerations with the additional assumptions that there is only a single scalar state variable ξ that takes on discrete values ξ^j , ($j = 0, 1 \dots$), and that the function $F(\epsilon, \xi^j)$ in the definition of $\Gamma(\xi^j)$ be identified with level surfaces of the dissipation function $\phi(\epsilon)$. A state transition takes place each time the material switches from an inelastic to an elastic domain. The state variable ξ^0 is defined by

$$\xi^0 = 0 \tag{4}$$

for material in the initial state, and by

$$\xi^j = \max \phi(\epsilon), \quad j = 1, 2, \dots \tag{5}$$

otherwise, where $\max \phi(\epsilon)$ represents the maximum value of energy dissipation encountered in the previous inelastic domain. Also, the function $F(\epsilon, \xi^j)$ is taken in the form

$$F(\epsilon, \xi^j) = \phi(\epsilon) - \xi^j. \tag{6}$$

We assert that the state variable ξ is monotonically increasing, which implies that the threshold surface $\Gamma(\xi)$ always expands in strain space in the sense that the elastic domain interior to the surface $\Gamma(\xi^j)$ must always be included in the elastic domain interior to $\Gamma(\xi^{j+1})$. Also, the parameters that occur in the description of the constitutive behavior in both the elastic and inelastic domains are functions of the state variable ξ^j , thereby implying the existence of a denumerable set of elastic and inelastic domains parametrized by ξ^j alone. This leads to the important conclusion that the only memory the material has of its past is via the state variable ξ^j , which implies a one parameter type of history dependence.

Some aspects of the inelastic behavior implied by this model are illustrated in Fig. 2, which shows how the dissipation density function and an arbitrary stress component σ_w vary with ϵ along three arbitrary radial paths in strain space in the two inelastic domains bounded by the surfaces Γ^0 and Γ^1 , respectively. Here Γ^j denotes $\Gamma(\xi^j)$ and increasing values of j imply increasing values of the associated state variable ξ^j .

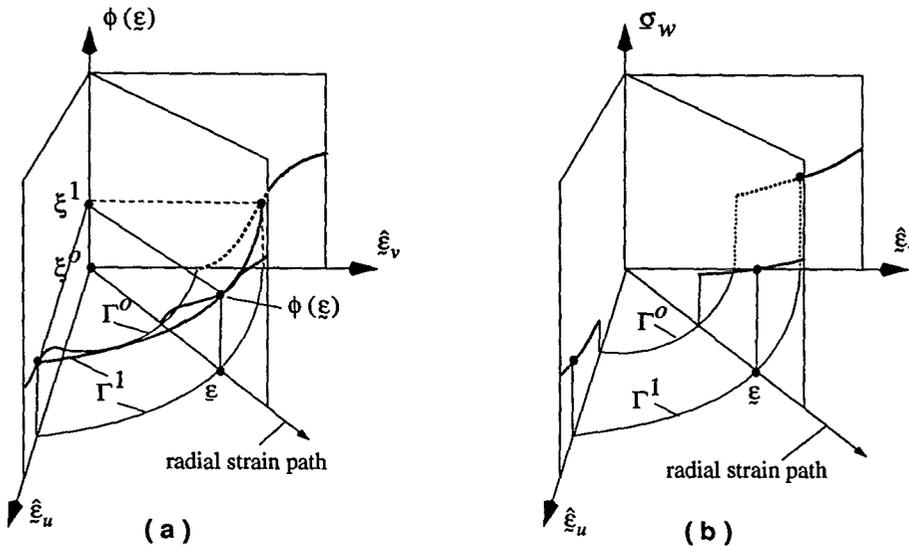


Fig. 2. Dissipated energy density as a function of (a) strains, and (b) corresponding stress–strain space.

3.1. Inelastic behavior

Consider the inelastic behavior regime first, and assume that the work potential $\psi(\boldsymbol{\varepsilon})$ can be taken as

$$\psi(\boldsymbol{\varepsilon}) = \Phi(\boldsymbol{\varepsilon}) + \phi(\boldsymbol{\varepsilon}), \quad (7)$$

where

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}(\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon}). \quad (8)$$

This particular form for Eq. (8) is chosen because the term $\Phi(\boldsymbol{\varepsilon})$ is equal to the energy density recovered if the material were to unload linearly from its present state, and, as is shown in this section, is energetically consistent with the postulated elastic unloading behavior. It may not be the only choice, but it is by far the simplest.

Since $\boldsymbol{\sigma} = \text{grad}_{\boldsymbol{\varepsilon}} \psi(\boldsymbol{\varepsilon})$, by definition, taking the gradient of Eq. (7) gives

$$\boldsymbol{\sigma} = \text{grad}_{\boldsymbol{\varepsilon}} \Phi(\boldsymbol{\varepsilon}) + \text{grad}_{\boldsymbol{\varepsilon}} \phi(\boldsymbol{\varepsilon}). \quad (9)$$

Eq. (9) is a system of first order linear partial differential equations in $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ and can be dealt with routinely the method of characteristics [11] when the proper initial conditions are specified on the surface $\Gamma(\xi^j)$; namely,

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{\Gamma^j}) = [C^e(\boldsymbol{\varepsilon}^{\Gamma^j}, (\alpha_1, \alpha_2))]. \quad (10)$$

Here (α_1, α_2) are surface coordinates on $\Gamma(\xi^j)$, $\boldsymbol{\varepsilon}^{\Gamma^j} = \boldsymbol{\varepsilon}^{\Gamma(\xi^j, \alpha_1, \alpha_2)}$ denotes the set of $\boldsymbol{\varepsilon} \in \Gamma(\xi^j)$ and are regarded as specified, and $[C^e(\boldsymbol{\varepsilon}^{\Gamma^j}, (\alpha_1, \alpha_2))]$ is the known matrix representation of the constitutive relation for the previous elastic domain.

In general, explicit closed-form expressions for $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ cannot be formed that satisfy the initial conditions of Eq. (10). There is, however, one such case where such a solution can indeed be found; namely the case where the constitutive relation for the prior elastic domain, i.e., Eq. (10) is linear in $\boldsymbol{\varepsilon}$, i.e.,

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = [C^e(\boldsymbol{\varepsilon}^{\Gamma^j}, (\alpha_1, \alpha_2))] \boldsymbol{\varepsilon}. \quad (11)$$

and where the matrix $[C^e(\boldsymbol{\varepsilon}^{\Gamma^j}, (\alpha_1, \alpha_2))]$ is constant over the transition surface $\Gamma(\xi^j)$, i.e., $[C^e(\boldsymbol{\varepsilon}^{\Gamma^j}, (\alpha_1, \alpha_2))] = [C^e(\xi^j)]$ on Γ^j . These criteria are certainly met when the elastic domain corresponds to that of the "virgin" material as-

suming it to be linear and homogeneous in a mechanical sense. Thus, taking the scalar product of Eq. (9) with $\boldsymbol{\varepsilon}$ and using the definition of $\Phi(\boldsymbol{\varepsilon})$ decouples the system of Eq. (9) and results in the following single first order linear partial differential equation for $\Phi(\boldsymbol{\varepsilon})$:

$$\boldsymbol{\varepsilon} \cdot \text{grad}_{\boldsymbol{\varepsilon}} \Phi - 2\Phi = -(\boldsymbol{\varepsilon} \cdot \text{grad}_{\boldsymbol{\varepsilon}} \phi(\boldsymbol{\varepsilon})), \quad (12)$$

with initial conditions

$$\begin{aligned} \Phi(\boldsymbol{\varepsilon}^{\Gamma^j}) = & \frac{1}{2} \left\{ \boldsymbol{\varepsilon}^{\Gamma^j}(\xi^j, \alpha_1, \alpha_2) [C^e(\xi^j, \alpha_1, \alpha_2)] \right. \\ & \left. \times \boldsymbol{\varepsilon}^{\Gamma^j}(\xi^j, \alpha_1, \alpha_2) \right\}. \end{aligned} \quad (13)$$

Eq. (12) with initial conditions of Eq. (13) again can be solved by means of characteristics to yield the following expression for $\Phi(\boldsymbol{\varepsilon})$:

$$\begin{aligned} \Phi(\xi^j, \alpha_1, \alpha_2, \beta) \\ = \beta^2 \left(\Phi(\boldsymbol{\varepsilon}^{\Gamma^j}) - \int_1^\beta \frac{1}{\zeta^2} \frac{d\phi}{d\zeta} d\zeta \right), \end{aligned} \quad (14)$$

evaluated on the characteristic lines. The above integral is taken along a characteristic line, that is defined by

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\xi^j, \alpha_1, \alpha_2, \beta) = \boldsymbol{\varepsilon}^{\Gamma^j}(\xi^j, \alpha_1, \alpha_2)\beta, \quad (15)$$

and where by $d\phi/d\zeta$ it is meant that

$$\frac{d\phi}{d\zeta} = \frac{d}{d\zeta} \phi(\zeta \boldsymbol{\varepsilon}^{\Gamma^j}(\xi^j, \alpha_1, \alpha_2)). \quad (16)$$

Eq. (15) can be regarded as a coordinate transformation relating the coordinates $(\alpha_1, \alpha_2, \beta)$ to $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. To obtain $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ we also need the inverse transformation $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \rightarrow (\alpha_1, \alpha_2, \beta)$ and we must be able to evaluate the gradients $\text{grad}_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}^{\Gamma^j}(\boldsymbol{\varepsilon})$, $\text{grad}_{\boldsymbol{\varepsilon}} \beta^2(\boldsymbol{\varepsilon})$, and $\text{grad}_{\boldsymbol{\varepsilon}} \boldsymbol{\alpha}(\boldsymbol{\varepsilon})$, with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. A closed-form expression for the inverse can be found when the α_γ ($\gamma = 1, 2$) are taken as the cosines of the angles that the position vector $|\boldsymbol{\varepsilon}^{\Gamma^j}|$ makes with the $\boldsymbol{\varepsilon}_\gamma$ axes, i.e.,

$$\alpha_\gamma = \boldsymbol{\varepsilon}_\gamma^{\Gamma^j} / |\boldsymbol{\varepsilon}^{\Gamma^j}|. \quad (17)$$

Then, using Eq. (15), Eq. (17) becomes

$$\alpha_\gamma = \boldsymbol{\varepsilon}_\gamma / |\boldsymbol{\varepsilon}|, \quad (18)$$

and from Eq. (15), there results for β :

$$\beta = |\boldsymbol{\varepsilon}| / |\boldsymbol{\varepsilon}^{\Gamma^j}|, \quad (19)$$

where the term ϵ^{Γ^j} in Eq. (19) is now understood to be a function of ϵ whose values are obtained by substituting Eq. (18) into $\epsilon^{\Gamma^j}(\xi^j, \alpha_1, \alpha_2)$. Eqs. (18) and (19) therefore define the sought after coordinate transformation $(\epsilon_1, \epsilon_2, \epsilon_3) \rightarrow (\alpha_1, \alpha_2, \beta)$.

The gradients $[\text{grad}_\epsilon \alpha(\epsilon)]$, $\text{grad}_\epsilon \epsilon^{\Gamma^j}(\epsilon)$, and $\text{grad}_\epsilon \beta^2(\epsilon)$ are now readily evaluated by using Eqs. (15) and (17) through (19). Thus,

$$\text{grad}_\epsilon \alpha_\gamma(\epsilon) = \frac{1}{|\epsilon|} \left(\delta_{i\gamma} - \frac{\epsilon_i \epsilon_\gamma}{|\epsilon|^2} \right), \quad (20)$$

where

$$\delta_{i\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (21)$$

$$[\text{grad}_\epsilon \epsilon^{\Gamma^j}(\epsilon)] = [\text{grad}_\alpha \epsilon^{\Gamma^j}(\alpha)] [\text{grad}_\epsilon \alpha(\epsilon)], \quad (22)$$

where $[\text{grad}_\alpha \epsilon^{\Gamma^j}(\alpha)] = [\text{grad}_\alpha \epsilon^{\Gamma^j}(\alpha)]|_{\alpha=\alpha(\epsilon)}$ is

known since the transition surface is already determined, and

$$\text{grad}_\epsilon \beta^2(\epsilon) = \left(\frac{2}{|\epsilon^{\Gamma^j}|^2} \left(\mathbf{I} - \beta [\text{grad}_\epsilon \epsilon^{\Gamma^j}(\epsilon)]^T \right) \right) \epsilon, \quad (23)$$

where \mathbf{I} is the identity matrix.

The stress field $\sigma(\epsilon)$ can now be obtained by substituting the expression for $\Phi(\xi, \alpha_1, \alpha_2, \beta)$ in Eq. (14) into Eq. (9) and using the relationships between ϵ and $(\xi, \alpha_1, \alpha_2, \beta)$ as given in Eq. (15). Thus $\sigma(\epsilon)$ is given by

$$\begin{aligned} \sigma = & \left([C^e] + \frac{2}{|\epsilon^{\Gamma^j}|^2} \left(\mathbf{I} - \beta [\text{grad}_\epsilon \epsilon^{\Gamma^j}(\epsilon)]^T \right) \right) \\ & \times \int_1^\beta \frac{1}{\zeta^2} \frac{d\phi}{d\zeta} d\zeta \epsilon \\ & - \beta^2 \int_1^\beta \frac{1}{\zeta^2} [\text{grad}_\epsilon \epsilon^{\Gamma^j}(\epsilon)] \text{grad}_\alpha \left(\frac{d\phi}{d\zeta} \right) d\zeta. \end{aligned} \quad (24)$$

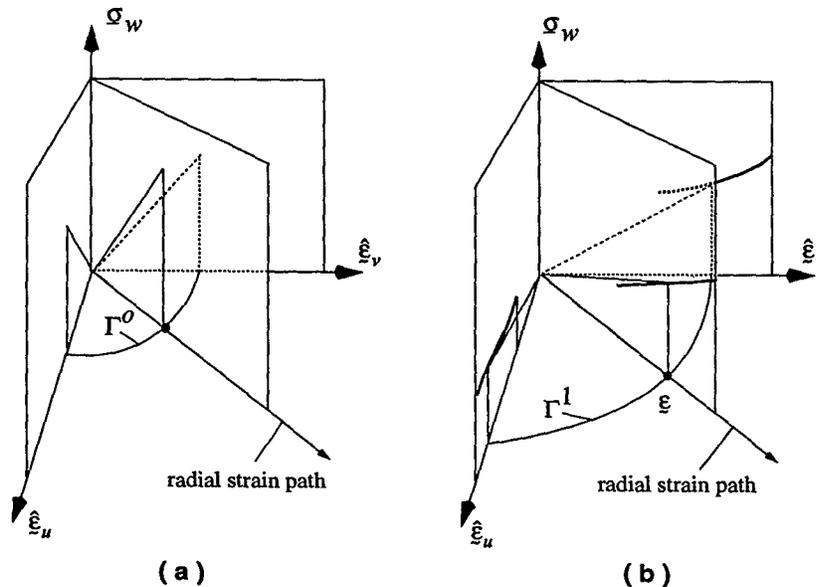


Fig. 3. Stress-strain behavior for state Γ^0 (a) and state Γ^1 (b).

3.2. Elastic behavior

In the elastic regime we assume that the material can be modeled as a particular type of nonlinear hyperelastic material; namely one that allows for no initial stresses and has linear stress-strain behavior along radial paths in strain space. This later assumption was invoked to maintain energetic consistency with the representation for ψ in Eq. (7) so that the energy available for elastic unloading is always given by $\Phi(\boldsymbol{\varepsilon})$. This assumption also agrees with repeated observations of linear unloading behavior during the IPL tests.

The type of behavior we are attempting to model is illustrated in Fig. 3 where the linear behavior along radial paths is shown for two elastic domains with associated transition surfaces Γ^0 and Γ^1 (Fig. 3(a) and 3(b), respectively). The important point to note here is that the elastic constitutive behavior is generally nonlinear, it is only guaranteed to be linear along the radial paths mentioned above. Also note that the effective elastic stiffness is expected to decrease with increasing values of the state variable ξ^j , which in turn correspond to increasing values of energy dissipation as illustrated in the Fig. 3.

Consistent with the above assumptions, we propose a constitutive relation in the elastic domain having the form

$$\boldsymbol{\sigma} = (C^e(\xi, \alpha_1, \alpha_2)) |\boldsymbol{\varepsilon}|, \quad (25)$$

where C^e is now taken to be a vector rather than a matrix. Eq. (25) has a deceptively simple form; however, it is really quite general and easily subsumes the usual linear form

$$\boldsymbol{\sigma} = [A]\boldsymbol{\varepsilon}. \quad (26)$$

That this is indeed the case is readily demonstrated since $\boldsymbol{\varepsilon}$ can always be written as $\boldsymbol{\varepsilon} = \hat{\boldsymbol{\varepsilon}}(\alpha_1, \alpha_2) |\boldsymbol{\varepsilon}|$ along some path with $\hat{\boldsymbol{\varepsilon}}$ being the associated unit vector. Therefore Eq. (26) assumes the form of Eq. (25) with C^e given as

$$C^e(\alpha_1, \alpha_2) = [A]\hat{\boldsymbol{\varepsilon}}(\alpha_1, \alpha_2). \quad (27)$$

For the initial state ξ_0 , the constitutive behavior is assumed to be given by the equivalent form of Eq. (26) with the matrix A computed using the

“virgin” material properties. For subsequent states ξ_i , $i = 1, 2, \dots$, Eq. (25) is used where stress continuity across the surface Γ^j leads to the following expression for the vector $C^e(\xi^j, \boldsymbol{\varepsilon})$:

$$C^e(\xi^{j+1}, \boldsymbol{\varepsilon}) = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{\Gamma^j}) / |\boldsymbol{\varepsilon}^{\Gamma^j}|, \quad (28)$$

where $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{\Gamma^j})$ is computed by evaluating the stresses given by Eq. (19) on the surface Γ^j .

It is emphasized that Eq. (25) with C^e given by Eq. (28) usually results in elastic constitutive behavior which is inherently nonlinear because $C_s^e(\xi, \boldsymbol{\varepsilon})$ is generally not of the form given in Eq. (26).

3.3. Determination of the parameter vector

The irreversible portion R^i of the total energy imparted to the specimen via the boundary loads at every observation point i in an IPL test series for a given material can be computed by numerical integration since all tractions t and applied boundary displacements u required for the computations were previously measured and stored. Thus, R^i is given by

$$R^i = \int_0^{a''} f \cdot du - \frac{1}{2}(f^i \cdot u^i), \quad (29)$$

where $i = 1, \dots, n$. On the other hand, the energy absorbed by the specimen is by linear additivity

$$\Phi^i = \int_{\text{vol}} \varphi(\boldsymbol{\varepsilon}(x)) dx = c_j \int_{\text{vol}} \chi_j(\boldsymbol{\varepsilon}^i(c, x)) dx, \quad (30)$$

where the volume integration is to be taken in a Stieltjes sense over the plies. When doing ply-level analyses of angle-ply materials, we have used the representation given by Eq. (3), and the summation convention is in effect. But Φ^i must equal R^i by energy conservation, therefore

$$R^i = c_j \int_{\text{vol}} \chi_j(\boldsymbol{\varepsilon}^i(c, x)) dx. \quad (31)$$

The quantity $\boldsymbol{\varepsilon}^i(c, x)$ that appears in Eq. (30) is the specimen strain field corresponding to the i th observation and is not known explicitly; however, it must satisfy the constitutive relation as given by Eq. (24) as well as the equilibrium and strain-displacement equations, i.e.

$$\text{div } \boldsymbol{\sigma}^i = 0, \quad (32)$$

and

$$\epsilon^i = \text{sym}(\text{grad } u^i). \tag{33}$$

The appropriate boundary conditions are free surface conditions (no tractions) except on the gripped surfaces where displacements are prescribed. Eqs. (29) through (33) form a coupled system of nonlinear equations whose solution places restrictions on the quantities (c , ϵ , σ , u).

4. Method of solution

The nature of the system of Eqs. (29) through (33) suggests a solution by the following iterative procedure:

1) An initial estimate of the strain field, i.e., $\epsilon^{i0}(x)$ is obtained by solving Eqs. (32) and (33) using the constitutive relation

$$\sigma = [C^0]\epsilon, \tag{34}$$

where $[C^0]$ is the stiffness matrix for the unstressed material and can either be obtained from the literature (when available) or estimated from mixture theory. The problem defined here is linear and is readily solved by finite element methods, or, in certain cases, by closed-form procedures [12–14].

2) Using $\epsilon^{i0}(x)$ as the estimate for $\epsilon^i(\alpha, x)$, an estimate of the parameter vector c is obtained from Eq. (31) subject to the constraints that each component of the imparted energy be monotonically increasing along every loading path. This involves solving a linear or quadratic programming problem as explained in Section 5.3 of Ref. [4].

3) Eqs. (32), (33), and (24) are solved numerically with the value of c in the representation for ϕ (Eq. (3)) taken as c^0 ; the new estimate of ϵ is utilized in Eq. (31) to obtain an updated estimate of c ; and the process is repeated until successive estimates differ by (hopefully) a sufficiently small amount.

Note that except for the initial step, determining the estimate of $\epsilon^i(c, x)$ always involves solving a set of nonlinear boundary value problems. Although this is in general difficult and time consuming, as a practical matter, unless one is interested in venturing deeply into the nonlinear

regime where significant strain redistribution occurs due to spatially widespread material stiffness changes, sufficient accuracy should be attainable by implementing only a single cycle of the iteration procedure. Thus, only the linear problem described in Item 1 above needs to be solved, and the difficulties mentioned above should be of little concern. If the need does arise to implement more than one cycle of the iteration procedure, techniques such as homotopic continuation methods [15] applied along each loading path are available to help alleviate the difficulties.

4.1. Initial estimates of the parameter vectors

As mentioned previously, $\epsilon^{i0}(x)$ is computed by solving a linear boundary value problem. Indeed, using the fact that any boundary displacement u produced by the IPL can be represented by a linear combination of opening/closing (u_1), sliding (u_0), and rotating (u_2) displacements, only three finite element analyses need be performed in this step since any strain field $\epsilon^{i0}(x)$ can be represented as

$$\epsilon^{i0} = \sum_k \epsilon^{i0k} \tag{35}$$

by linear superposition, where $\epsilon^{i0k}(x)$ are the strain fields corresponding to unit boundary displacements in the k th direction.

With $\epsilon^{i0}(x)$ known, the quantity $\int_{\text{vol}} \chi_j(\epsilon^{i0}(x)) dx$ in Eq. (31) may be computed. Thus, setting

$$\int_{\text{vol}} \chi_j(\epsilon^{i0}(x)) dx \equiv F_{ij}, \tag{36}$$

Eq. (31) may be written as the linear set of equations

$$[F]z = W, \tag{37}$$

where $W \equiv (R^1, \dots, R^n)^T$; $z \equiv c^0$, and

$$[F] = \begin{bmatrix} F_{11} & \dots & F_{1m} \\ \dots & \dots & \dots \\ F_{n1} & \dots & F_{nm} \end{bmatrix}, \tag{38}$$

as long as $\text{rank}([F]) < \text{rank}([\bar{F}])$, where: $[\bar{F}]$ is the augmented matrix

$$\begin{bmatrix} F_{11} & \dots & F_{1m} & W_1 \\ \dots & \dots & \dots & \dots \\ F_{n1} & \dots & F_{nm} & W_m \end{bmatrix}. \tag{39}$$

Eq. (37) is an overdetermined system and there exists no value of z that satisfies it exactly. Rather, one must seek to minimize the norm of the error vector $e = (W - [F]z)$ to obtain a best approximation to the solution. This is the usual state of affairs in parameter estimation; indeed, the more highly overdetermined the system, the better the situation since one generally wants to work with as many independent observations as possible to filter out any undesirable stochastic variations (noise). Only if enough observations are linearly dependent can it happen that $\text{rank}([F]) = \text{rank}([\bar{F}]) < m$, in which case there exists $n - \text{rank}([F])$ solutions to Eq. (37). This situation should not arise if the experimental testing program is well thought out.

Assuming for the moment that Eq. (37) is indeed overdetermined, we seek to minimize the norm of $e = W - [F]z$ subject to the constraints that

– the values of φ are positive at the node points of the interpolation mesh, i.e.,

$$z^i > 0, \quad (40)$$

to maintain positivity,

– the dissipated energy is monotonically increasing along each radial loading path in displacement space, i.e.,

$$\begin{aligned} \text{row}([F])_{i_j} \cdot z &> 0 \\ (\text{row}([F])_{i_{j+1}} - \text{row}([F])_{i_j}) \cdot z &\geq 0, \\ i &= i_p, \dots, i_q - 1, \end{aligned} \quad (41)$$

where Eq. (41) applies to each loading path, i_p is a point at the beginning of the path, and i_q is the end point.

Note that the above constraints are linear inequality constraints in z , and as such are readily dealt with. Two widely used measures of vector magnitude are the L^2 (Euclidean) and L^∞ (Chebychev) norms given by

$$\|x\|_{L^2}^2 = (x_1^2 + x_2^2 + \dots + x_n^2)^2, \quad (42)$$

and

$$\|x\|_{L^\infty} = \max_x (|x_1|, |x_2|, \dots, |x_n|) \quad (43)$$

respectively. Minimizing $\|e\|$ in Eq. (37) under the L^2 norm subject to the constraints of Eqs.

(40) and (41) is a problem in quadratic programming since the objective function $\|W - [F]z\|_{L^2}$ is quadratic in the variables (z_1, z_2, \dots, z_n) . This is a well-established discipline and many efficient procedures are available to deal with such problems. Minimizing $\|e\|$ under the L^∞ norm is just as straightforward, and perhaps easier, because there exists a well-known technique due to Lagrange for converting such problems into linear programming problems that can be solved using the simplex method or one of its variants. A highly readable discussion of these optimization issues is presented in [15,16].

5. Conclusions

A general theory for deriving the constitutive behavior of damaged composites is presented. It uses the full-scale nonlinear behavior such that the determination of the dissipation density function can be achieved by using the boundary force and displacement data obtained from the IPL tests.

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